Topic - 3.1 Notation, Sequences, Summations - Day 1

Sequences

Many creations in nature involve intricate mathematical designs, including a variety of spirals. For example, the arrangement of the individual florets in the head of a sunflower forms spirals. In some species, there are 21 spirals in the clockwise direction and 34 in the counterclockwise direction. The precise numbers depend on the species of sunflower: 21 and 34, or 34 and 55, or 55 and 89, or even 89 and 144.

This observation becomes even more interesting when we consider a sequence of numbers investigated by Leonardo of Pisa, also known as Fibonacci, an Italian mathematician of the thirteenth century. The **Fibonacci sequence** of numbers is an infinite sequence that begins as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots.$$

HW 3.1 Day 1: p.1010 - *printout*:
#'s: 1, 9, 11, 17, 20, 23, 25, 31, 34, 40

The first two terms are 1. Every term thereafter is the sum of the two preceding terms. For example, the third term, 2, is the sum of the first and second terms: $1 + 1 = 2$. The fourth term, 3, is the sum of the second and third terms: $1 + 2 = 3$, and so on. Did you know that the number of spirals in a daisy or a sunflower, 21 and 34, are two Fibonacci numbers? The number of spirals in a pine cone, 8 and 13, and a pineapple, 8 and 13, are also Fibonacci numbers.

We can think of the Fibonacci sequence as a function. The terms of the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

are the range values for a function f whose domain is the set of positive integers.

Domain:	1,	2,	3,	4,	5,	6,	7,	…
	↓	↓	↓	↓	↓	↓	↓	
Range:	1,	1,	2,	3,	5,	8,	13,	…

Thus, $f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 3, f(5) = 5, f(6) = 8, f(7) = 13,$ and so on.

The letter a with a subscript is used to represent function values of a sequence, rather than the usual function notation. The subscripts make up the domain of the sequence and they identify the location of a term. Thus, a_1 represents the first term of the sequence, a_2 represents the second term, a_3 the third term, and so on. This notation is shown for the first six terms of the Fibonacci sequence:

1,	1,	2,	3,	5,	8.
$a_1 = 1$	$a_2 = 1$	$a_3 = 2$	$a_4 = 3$	$a_5 = 5$	$a_6 = 8$

The notation a_n represents the nth term, or **general term**, of a sequence. The entire sequence is represented by $\{a_n\}$.

Definition of a Sequence

An **infinite sequence** $\{a_n\}$ is a function whose domain is the set of positive integers. The function values, or **terms**, of the sequence are represented by

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

Sequences whose domains consist only of the first n positive integers are called **finite sequences**.

EXAMPLE 1 Writing Terms of a Sequence from the General Term

Write the first four terms of the sequence whose nth term, or general term, is given:

a. $a_n = 3n + 4$

$a_1 = 3(1)+4 = 7$

$a_2 = 3(2)+4 = 10$

$a_3 = 3(3)+4 = 13$

$a_4 = 3(4)+4 = 16$

$\boxed{7, 10, 13, 16, \ldots}$

b. $a_n = \dfrac{(-1)^n}{3^n - 1}$.

$a_1 = \dfrac{(-1)^1}{3^1 - 1} = \boxed{\dfrac{-1}{2}}$

$a_2 = \dfrac{(-1)^2}{3^2 - 1} = \boxed{\dfrac{1}{8}}$

$a_3 = \dfrac{(-1)^3}{3^3 - 1} = \boxed{\dfrac{-1}{26}}$

$a_4 = \dfrac{(-1)^4}{3^4 - 1} = \boxed{\dfrac{1}{80}}$

Recursion Formulas

In Example 1, the formulas used for the nth term of a sequence expressed the term as a function of n, the number of the term. Sequences can also be defined using **recursion formulas**. A recursion formula defines the nth term of a sequence as a function of the previous term. Our next example illustrates that if the first term of a sequence is known, then the recursion formula can be used to determine the remaining terms.

EXAMPLE 2 Using a Recursion Formula

Find the first four terms of the sequence in which $\boxed{a_1 = 5}$ and $\boxed{a_n = 3a_{n-1} + 2}$ for $n \geq 2$.

$a_1 = \boxed{5}$

$a_2 = 3(5)+2 = \boxed{17}$

$a_3 = 3(17)+2 = \boxed{53}$

$a_4 = 3(53)+2 = \boxed{161}$

Factorial Notation

Products of consecutive positive integers occur quite often in sequences. These products can be expressed in a special notation, called **factorial notation.**

Factorial Notation

If n is a positive integer, the notation $n!$ (read "n factorial") is the product of all positive integers from n down through 1.

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

0! (zero factorial), by definition, is 1.

$$\boxed{0! = 1} \;*$$

The values of $n!$ for the first six positive integers are

$$1! = 1$$
$$2! = 2 \cdot 1 = 2$$
$$3! = 3 \cdot 2 \cdot 1 = 6$$
$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$
$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

EXAMPLE 3 Finding Terms of a Sequence Involving Factorials

Write the first four terms of the sequence whose nth term is

$$a_n = \frac{20}{(n+1)!}$$

$$a_1 = \frac{20}{(1+1)!} = \frac{20}{2!} = \frac{20}{2} = \boxed{10}$$

$$a_2 = \frac{20}{(2+1)!} = \frac{20}{3!} = \frac{20}{6} = \boxed{\frac{10}{3}}$$

$$a_3 = \frac{20}{(3+1)!} = \frac{20}{4!} = \frac{20}{24} = \boxed{\frac{5}{6}}$$

$$a_4 = \frac{20}{(4+1)!} = \frac{20}{5!} = \frac{20}{120} = \boxed{\frac{1}{6}}$$

Summation Notation

The sum of the first n terms of a sequence is represented by the **summation notation**

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n,$$

where i is the **index of summation**, n is the **upper limit of summation**, and 1 is the **lower limit of summation**.

Any letter can be used for the index of summation. The letters i, j, and k are used commonly. Furthermore, the lower limit of summation can be an integer other than 1.

When we write out a sum that is given in summation notation, we are **expanding the summation notation**. Example 5 shows how to do this.

EXAMPLE 5 Using Summation Notation

Expand and evaluate the sum:

a. $\sum_{i=1}^{6}(i^2+1)$

$i=1$: $(1^2+1) = 2$

$i=2$: $(2^2+1) = 5$

$i=3$: $(3^2+1) = 10$

$i=4$: $(4^2+1) = 17$

$i=5$: $(5^2+1) = 26$

$i=6$: $(6^2+1) = 37$

$\boxed{\text{Sum} = 97}$

EXAMPLE 5 Using Summation Notation

Expand and evaluate the sum:

b. $\sum_{k=4}^{7}[(-2)^k - 5]$

$k=4$: $(-2)^4 - 5 = 11$

$k=5$: $(-2)^5 - 5 = -37$

$k=6$: $(-2)^6 - 5 = 59$

$k=7$: $(-2)^7 - 5 = -133$

$\boxed{\text{sum} = -100}$

November 13, 2019

Answer Key to HW 3.1 - Day 1

HW Day1: p.1010 - printout:
#'s: 1, 9, 11, 17, 20, 23, 25, 31, 34, 40

1.) 5, 8, 11, 14

9.) 2/5, 2/3, 6/7, 1

11.) 1, -1/3, 1/7, -1/15

17.) 4, 11, 25, 53

20.) 2, 3/2, 8/3, 15/2

23.) 272

25.) 120

31.) Sum = 60

34.) Sum = -4

40.) Sum = -19/30