4.6 - Trigonometric Equations - Day 1

Trigonometric Equations and Their Solutions

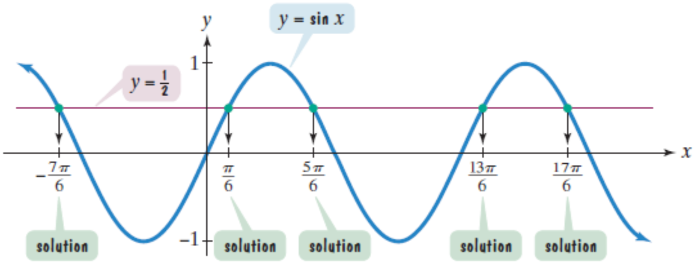
A **trigonometric equation** is an equation that contains a trigonometric expression with a variable, such as $\sin x$. We have seen that some trigonometric equations are identities, such as $\sin^2 x + \cos^2 x = 1$. These equations are true for every value of the variable for which the expressions are defined. In this section, we consider trigonometric equations that are true for only some values of the variable. The values that satisfy such an equation are its **solutions.** (There are trigonometric equations that have no solution.)

An example of a trigonometric equation is

$$\sin x = \frac{1}{2}.$$

A solution of this equation is $\frac{\pi}{6}$ because $\sin\frac{\pi}{6} = \frac{1}{2}$. By contrast, π is not a solution because $\sin \pi = 0 \neq \frac{1}{2}$.

Is $\frac{\pi}{6}$ the only solution of $\sin x = \frac{1}{2}$? The answer is no. Because of the periodic nature of the sine function, there are infinitely many values of x for which $\sin x = \frac{1}{2}$. **Figure 5.7** shows five of the solutions, including $\frac{\pi}{6}$, for $-\frac{3\pi}{2} \le x \le \frac{7\pi}{2}$. Notice that the x-coordinates of the points where the graph of $y = \sin x$ intersects the line $y = \frac{1}{2}$ are the solutions of the equation $\sin x = \frac{1}{2}$.

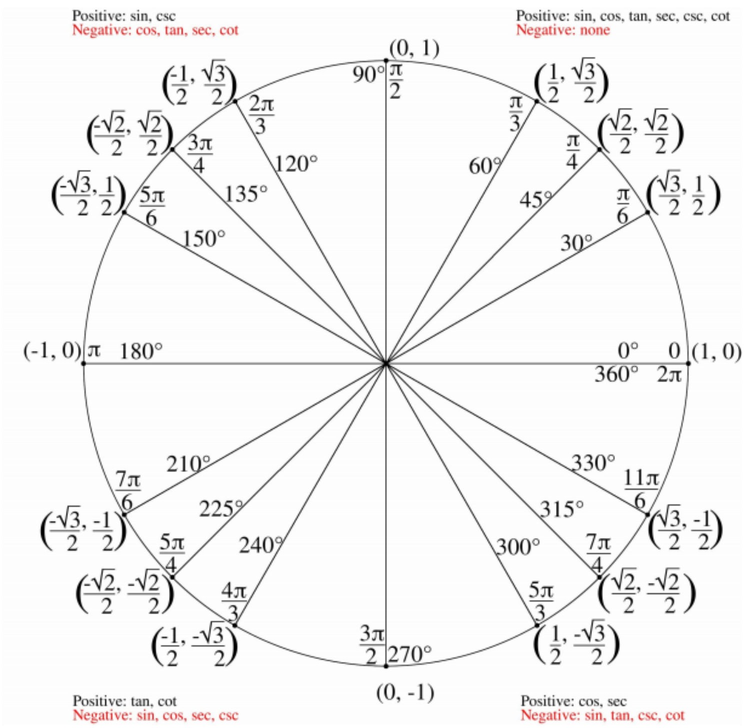


The Unit Circle

$\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \dfrac{y}{x}$



$\csc\theta = \dfrac{1}{y}$

$\sec\theta = \dfrac{1}{x}$

$\cot\theta = \dfrac{x}{y}$

Ex.1: Use substitution to determine whether the given x-value is a solution of the equation.

$\sin\frac{2x}{3} = -\frac{1}{2}$, where $x = \pi$

$\sin\frac{2\pi}{3} = -\frac{1}{2}$

"Does the y-value of the point at $\frac{2\pi}{3}$ equal $-\frac{1}{2}$?"

$\boxed{\text{False}}$

$\tan x = \sqrt{3}$, where $x = \frac{\pi}{3}$

$\tan\frac{\pi}{3} \overset{?}{=} \sqrt{3}$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
x y

$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$\frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\boxed{\text{TRUE}}$

Equations Involving a Single Trigonometric Function

To solve an equation containing a single trigonometric function:

- Isolate the function on one side of the equation.
- Solve for the variable.

EXAMPLE 1 Finding All Solutions of a Trigonometric Equation

$2\cos x + 1 = 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$

$\boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3}}$

"Where does the x-value (cos) equal $-\frac{1}{2}$?

$\sqrt{3}\tan x - 1 = 0$

$\sqrt{3}\tan x = 1$

$\tan x = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$\tan x = \frac{\sqrt{3}}{3}$

"Where does y/x equal $\frac{\sqrt{3}}{3}$?"

pt. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\boxed{x = \frac{\pi}{6}, \frac{7\pi}{6}}$

EXAMPLE 1 Finding All Solutions of a Trigonometric Equation

Solve the equation: $3 \sin x - 2 = 5 \sin x - 1.$

$-5\sin x \quad -5\sin x$

$-2\sin x - 2 = -1$

$+2 \quad +2$

$\frac{-2\sin x}{-2} = \frac{1}{-2}$

$\sin x = -\frac{1}{2}$

$y = -\frac{1}{2}$?

$\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

Check Point 1 Solve the equation: $5 \sin x = 3 \sin x + \sqrt{3}.$

$-3\sin x \quad -3\sin x$

$\frac{2\sin x}{2} = \frac{\sqrt{3}}{2}$

$\sin x = \frac{\sqrt{3}}{2}$

"$y = \frac{\sqrt{3}}{2}$ where ?"

$\boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}}$